A cube beyond three dimensions

Tesseract worksheet · use with the free interactive explorer

1. Count the structure

A tesseract has vertices (x, y, z, w), with each coordinate equal to -1 or +1. Join two vertices when exactly one coordinate differs.

Part	Count	Why
Vertices	16	2 × 2 × 2 × 2
Edges	32	16 vertices × 4 neighbors ÷ 2
Square faces	24	Choose 2 varying axes; fix the other 2
Cubic cells	8	Choose 1 fixed axis and its sign

2. Predict, then turn

Set XW = 0°, YW = 0° and choose Orthographic. Which distinct 4D vertices share a displayed position? What changes under Perspective?

Change XW to 90°. For the rotation $x' = x \cos\theta - w \sin\theta$, $w' = x \sin\theta + w \cos\theta$, what happens to (x, w)? Explain before checking.

3. Keep the distinction

Projection drops a coordinate from every point. A slice keeps only points at a chosen w. For the axis-aligned solid $[-1,1]^4$, both projection and the w = 0 slice are cubes, but they contain information from different sets of points.

Check: at 90°, (x, w) becomes (-w, x). In orthographic mode at zero angles, vertices differing only in w overlap. A crossing in the drawing need not be an intersection in 4D.

Source: Oliver Knill, Harvard Math 21b tesseract demonstration: people.math.harvard.edu/archive/21b_fall_03/4dcube/index.html

Slice a four-dimensional ball

Hypersphere worksheet · parent radius R = 1

1. One equation, three different outcomes

The solid 4-ball satisfies $x^2 + y^2 + z^2 + w^2 \leq 1$. At fixed w, its 3D slice has radius $r = \sqrt{(1 - w^2)}$ when $|w| \leq 1$. If $|w| > 1$, no slice exists.

For $|w| < 1$, slicing the filled 4-ball gives a solid 3-ball; slicing only its 3-sphere boundary gives a 2-sphere surface. At $w = \pm 1$, either section is a single point.

w	r	Section of solid 4-ball
-1.20	—	Empty
-1.00	0.00	Single point
-0.60	0.80	Solid 3-ball
0.00	1.00	Solid 3-ball
0.60	0.80	Solid 3-ball
1.00	0.00	Single point
1.20	—	Empty

2. Explain what you observe

Why do w = -0.60 and w = +0.60 give equal radii? What changes when you select the boundary instead of the filled object?

Check: the radius depends on w^2, so reversing the sign leaves it unchanged. The boundary slice includes the spherical surface; the filled slice also includes its interior.

These are standard Euclidean geometry calculations. They do not establish that a physical object or symbol has a four-dimensional origin.